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Second-Order Linear Recurrence Relations and Periodicity

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Abstract: A sequence, $\{s_n\}$, which follows a second-order linear recurrence relation satisfies $s_{n+1} = c_1 s_n + c_2 s_{n-1}$, for some constants c_1 and c_2 where $c_2 \neq 0$. For any positive integer k, we construct such a sequence with period k. By varying the initial values s_0 and s_1, a given second-order linear recurrence relation can generate at most three distinct non-trivial periods, one of which is the least common multiple of the other two.

1. Introduction

A sequence is a function $f(n) = s_n$ with domain $\mathbb{N} \cup \{0\}$. We will denote this sequence $\{s_n\}$. A sequence is said to follow a second-order linear recurrence relation if there exists two constants, c_1 and c_2, $c_2 \neq 0$ such that

$$s_n = c_1 s_{n-1} + c_2 s_{n-2}, n \geq 2.$$

A famous example of this is:

Example 1.1. *The Fibonacci Sequence:*

$$s_0 = 0$$
$$s_1 = 1$$
$$s_n = 1 \cdot s_{n-1} + 1 \cdot s_{n-2}.$$

Thus the sequence is

$$0, 1, 1, 2, 3, 5, 8, 13, 21, ...$$

Given a sequence which follows a second-order linear recurrence relation, we are interested in determining if that sequence is periodic. Here, we say that a sequence $\{s_n\}$ is periodic if there exists a positive integer k such that $s_{n+k} = s_n$ for all $n \geq 0$. This k is called the period of the sequence. We can see that the above example is not periodic but we can construct sequences that are.

Example 1.2.

$$s_0 = 0$$
$$s_1 = 1$$
$$s_n = \sqrt{3} s_{n-1} - s_{n-2}.$$

Thus $c_1 = \sqrt{3}$ and $c_2 = -1$. Then the sequence is

$$0, 1, \sqrt{3}, 2, \sqrt{3}, 1, 0, -1, -\sqrt{3}, -2, -\sqrt{3}, -1, 0, 1, \sqrt{3}, ...$$

The above example is indeed periodic with a period of 12.

Given these examples, for a sequence $\{s_n\}$ which follows a second order linear recurrence relation, some questions are:

- Is $\{s_n\}$ always periodic? We can already see that in general, the answer is no.
- What conditions must be satisfied in order for $\{s_n\}$ to be periodic?
- If $\{s_n\}$ is periodic, what are the possible periods?

In this paper we answer these questions and determine how to choose c_1, c_2, s_0 and s_1 such that we can create a periodic sequence.

2. Preliminaries

In this section we discuss some properties of periodic sequences and develop the linear algebra and complex numbers ideas needed in order to get our results.
For this section, we fix a sequence $\{s_n\}$ satisfying the second order linear recurrence relation $s_n = c_1 s_{n-1} + c_2 s_{n-2}$.

Lemma 2.1. *It is the case that $\{s_n\}$ is periodic if and only if there exists a $k > 0$ such that $s_0 = s_k$ and $s_1 = s_{k+1}$. Furthermore, the period of the sequence is the smallest k for which this occurs.*

Proof: Let $\{s_n\}$ be periodic. Recall that this means that there exists a positive integer k such that $s_{n+k} = s_n$ for all $n \geq 0$. Thus pick $n = 0$, and $n = 1$ then we have $s_k = s_0$ and $s_{k+1} = s_1$.

Let $s_k = s_0$ and $s_{k+1} = s_1$ for some $k > 0$. We want to show that $s_{n+k} = s_n$ for all $n \geq 0$, which we can do inductively. For the base case note that $s_2 = c_1 s_1 + c_2 s_0$ and $s_{k+2} = c_1 s_{k+1} + c_2 s_k$, and since $s_k = s_0$ and $s_{k+1} = s_1$, we have that $s_2 = s_{k+2}$. Assume $s_\ell = s_{k+\ell}$ and $s_{\ell+1} = s_{k+\ell+1}$. Now consider $s_{\ell+2} = c_1 s_{\ell+1} + c_2 s_\ell = c_1 s_{k+\ell+1} + c_2 s_{k+\ell} = s_{k+\ell+2}$. Thus by mathematical induction, $s_{n+k} = s_n$ for all $n \geq 0$.

That the period is the smallest k such that $s_0 = s_k$ and $s_1 = s_{k+1}$ should be clear. $\square$

Thus we simply need to find a k such that $s_0 = s_k$ and $s_1 = s_{k+1}$ in order to determine that $\{s_n\}$ is periodic. However, we can imagine having an $\{s_n\}$ such that the period is large, for example, 1000. Thus we would have that $s_0 = s_{1000}$ and $s_1 = s_{1001}$. However, in order find that the period was 1000, we would have had to compute $s_2, s_3, s_4, ..., s_{1001}$ and this is incredibly tedious. We can avoid this and gain a clearer insight into the problem by considering the information given by our second order linear recurrence relation and consecutive terms in our sequences as matrices and vectors respectively and using linear algebra techniques to help analyze periodicity.

We can associate a 2×2 matrix A to $\{s_n\}$, namely

$$A = \begin{bmatrix} 0 & 1 \\ c_2 & c_1 \end{bmatrix}.$$

Furthermore, let s_n be the n^{th} term in our sequence, and define

$$x_n = \begin{bmatrix} s_n \\ s_{n+1} \end{bmatrix}$$

for $n \geq 0$.

Note that

$$x_0 = \begin{bmatrix} s_0 \\ s_1 \end{bmatrix}$$

is the vector with our initial two terms.

Now consider what happens if we perform Ax_n. We have that

$$Ax_n = \begin{bmatrix} 0 & 1 \\ c_2 & c_1 \end{bmatrix} \begin{bmatrix} s_n \\ s_{n+1} \end{bmatrix} = \begin{bmatrix} s_{n+1} \\ c_1 s_{n+1} + c_2 s_n \end{bmatrix} = \begin{bmatrix} s_{n+1} \\ s_{n+2} \end{bmatrix} = x_{n+1}.$$

Note that this occurs by our recurrence relation. Thus we have that $Ax_n = x_{n+1}$, which gives us another method to compute terms in the sequence.

Of course, this does not appear to avoid the tediousness of multiple computations, and rather it seems that we have further complicated it. However we will see further on that this approach does indeed greatly simplify the situation.

Note that for a sequence to be periodic of period k it must be the case that $x_0 = x_k$ due to Lemma 2.1 and the definition of x_n, since then we have

$$x_0 = \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} s_k \\ s_{k+1} \end{bmatrix} = x_k.$$

Thus we want to find a simple way to compute x_k.

Lemma 2.2. *For all $n \geq 0, A^n x_0 = x_n$.*

Proof: We will prove this by induction on n.
Consider

$$Ax_0 = \begin{bmatrix} 0 & 1 \\ c_2 & c_1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} s_1 \\ c_1 s_1 + c_2 s_0 \end{bmatrix}.$$

Now note that

$$x_1 = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} s_1 \\ c_1 s_1 + c_2 s_0 \end{bmatrix},$$

and thus $Ax_0 = x_1$ proving the base case.

Assume that $A^m x_0 = x_m$ for some $m \geq 2$.

Now consider

$$A^{m+1} x_0 = AA^m x_0 = Ax_m = \begin{bmatrix} 0 & 1 \\ c_2 & c_1 \end{bmatrix} \begin{bmatrix} s_m \\ s_{m+1} \end{bmatrix} = \begin{bmatrix} s_{m+1} \\ c_1 s_{m+1} + c_2 s_m \end{bmatrix} = \begin{bmatrix} s_{m+1} \\ s_{m+2} \end{bmatrix} = x_{m+1}.$$

Thus by mathematical induction we have that $A^n x_0 = x_n$. □

In order to continue it is useful for us to consider the definitions and properties of eigenvalues and eigenvectors.

Definition 2.3. *An eigenvector for A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. This λ is called an eigenvalue for A.*

Eigenvectors and eigenvalues help us determine quickly when a sequence is periodic. Suppose x_0 is an eigenvector with eigenvalue λ. Then we have that

$$x_n = A^n x_0 = A^{n-1}Ax_0 = A^{n-1}\lambda x_0 = A^{n-2}\lambda^2 x_0 = \cdots = \lambda^n x_0.$$

This greatly simplifies the computation of x_n since we now have x_n in terms of x_0 with a constant multiplier to a power which is considerably easier to deal with than our previous approach. We now know that in order for the sequence to be periodic of period k, we simply need to find eigenvalues such that $\lambda^k = 1$ since then $x_k = \lambda^k x_0$ becomes $x_k = 1 \cdot x_0$ which satisfies the criterion of being periodic.

Thus $\{s_n\}$ is periodic if and only if there exists a k such that

$$x_k = \lambda^k x_0 = x_0.$$

Now we want to know how the coefficients c_1 and c_2 and the eigenvalues are related. First, λ is an eigenvalue if and only if $\det(A - \lambda I) = 0$. In other words,

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 0 & 1 \\ c_2 & c_1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} -\lambda & 1 \\ c_2 & c_1 - \lambda \end{bmatrix}\right) \\
&= -\lambda\,(c_1 - \lambda) - c_2 \\
&= \lambda^2 - \lambda c_1 - c_2 \\
&= 0.
\end{aligned}$$

Thus, λ is an eigenvalue if it is a solution to $x^2 - c_1 x - c_2 = 0$. We call this the characteristic equation for the recurrence relation. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be solutions to the characteristic equation. We have

$$(x - \lambda_1)(x - \lambda_2) = x^2 - (\lambda_1 + \lambda_2)x - (-\lambda_1\lambda_2) = 0.$$

Since

$$x^2 - c_1 x - c_2 = x^2 - (\lambda_1 + \lambda_2)x - (-\lambda_1\lambda_2) = 0,$$

we have $c_1 = \lambda_1 + \lambda_2$ and $c_2 = -\lambda_1\lambda_2$.

Roots of unity, described below, play an integral role in obtaining our results.

Definition 2.4. *The n^{th} roots of unity are the complex roots of the equation $z^n = 1$.*

Specifically, the n^{th} roots of unity can be written in rectangular form as $\cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n})$ or in polar form as $e^{\frac{2\pi k}{n}i}$ where $k = 0, 1, 2, ..., n-1$ by [1, p. 34].

Example 2.5. *The 2^{nd} roots of unity are the roots of $z^2 = 1$ which are clearly -1 and 1. But we can also show this using our formula:*

$$\begin{aligned}
\cos(0 \cdot \pi) + i\sin(0 \cdot \pi) &= 1 \\
\cos(1 \cdot \pi) + i\sin(1 \cdot \pi) &= -1.
\end{aligned}$$

Example 2.6. *The* 6^{th} *roots of unity are* $\cos(\frac{2\pi k}{6}) + i\sin(\frac{2\pi k}{6})$, $0 \le k \le 5$, *and are*

$$1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Notice that -1 is both a second root of unity and a sixth root of unity.

Definition 2.7. *A number* $r \in \mathbb{C}$ *is a primitive* n^{th} *root of unity if* n *is the smallest positive integer of* m *for which* $r^m = 1$.

Thus, -1 is a sixth root of unity but a primitive second root of unity.

The reader can check that an n^{th} root of unity $\cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n})$ is primitive if and only if $\gcd(k, n) = 1$; also, if z is a primitive n^{th} root of unity, then $z^m = 1$ if and only if $n|m$.

3. Results

Theorem 3.1. *Let* $k \in \mathbb{N}$ *be given, and let*

$$\begin{aligned} s_0 &= 0 \\ s_1 &= 1 \\ s_n &= 2\cos\left(\frac{2\pi}{k}\right)s_{n-1} - s_{n-2}. \end{aligned}$$

Then the sequence has period k.

The proof of this theorem will be given later on with our other two results. (Proof omitted.)

Example 3.2. *Let* $k = 12$. *Let*

$$\begin{aligned} s_0 &= 0 \\ s_1 &= 1 \\ s_n &= 2\cos\left(\frac{2\pi}{12}\right)s_{n-1} - s_{n-2}. \end{aligned}$$

The terms of the sequence are:

$$0, 1, \sqrt{3}, 2, \sqrt{3}, 1, 0, -1, -\sqrt{3}, -2, -\sqrt{3}, -1, 0, 1, \ldots.$$

Thus we can see that the period is 12 and this agrees with Theorem 3.1.

Theorem 3.3. *Let* λ *be a primitive* k^{th} *root of unity which is a solution to the characteristic equation. If* $s_1/s_0 = \lambda$, *then the period of the sequence is* k.

Proof: Recall that a solution to the characteristic equation is an eigenvalue for A. Let v be an eigenvector with eigenvalue λ, and let x_0 be a vector of initial conditions. We have

$$x_k = A^k x_0 = \lambda^k x_0 = x_0.$$

Recall that k is a primitive root of unity. If $x_n = x_0$, then $k \mid n$, hence $k \le n$ and the period is k.

$\square$

Example 3.4. *Let* $\lambda_1 = i$ *and* $\lambda_2 = 1$. *Then* $c_1 = \lambda_1 + \lambda_2 = i + 1$ *and* $c_2 = -\lambda_1\lambda_2 = -i$, *so the recurrence relation we obtain from these eigenvalues is*

$$s_n = (i+1)s_{n-1} - is_{n-2}.$$

Pick $s_0 = 1$ *and* $s_1 = i$, *then clearly* $s_1/s_0 = i$. *Thus the terms of the sequence are :*

$$1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \ldots.$$

This sequence has period 4, *as expected, since* $\lambda_1 = i$ *is a primitive* 4^{th} *root of unity.*

Example 3.5. *Let $\{s_n\}$ be as above, and let $s_0 = s_1 = 3$. Then $s_1/s_0 = 1 = \lambda_2$, thus the terms of the sequence are*

$$3, 3, 3, 3 \ldots.$$

This sequence has period 1, as expected since 1 is a primitive 1^{st} root of unity.

Theorem 3.6. *Let λ_1 be a primitive m^{th} root of unity and let λ_1 be a primitive ℓ^{th} root of unity, and suppose both λ_1 and λ_2 are solutions to the characteristic equation. If $s_1/s_0 \neq \lambda_1,\ \lambda_2$, then the period of the sequence is $\operatorname{lcm}(m, \ell)$.*

Proof: Let u be an eigenvector with eigenvalue λ_1 and v be an eigenvector with eigenvalue λ_2, where λ_1 is a primitive m^{th} root of unity and λ_2 is a primitive ℓ^{th} root of unity.

Let x_0 be the vector of the initial conditions. Since u and v span $\mathbb{C}^2$ [2], we can write $x_0 = au + bv$, where a and b are both non-zero constants. Since $A^n x_0 = \lambda^n x_0$ for all n, we have $x_n = \lambda_1^n au + \lambda_2^n bv$. In order for

$$x_k = \lambda_1^k av + \lambda_2^k bu = au + bv = x_0,$$

we need $\lambda_1^k = \lambda_2^k = 1$, by the linear independence of u and v. This only occurs when $m \mid n$ and $\ell \mid n$, and the smallest such n is $\operatorname{lcm}(m, \ell)$, hence the period k is $\operatorname{lcm}(m, \ell)$.

□

Example 3.7. *Let $\lambda_1 = i$ and $\lambda_2 = 1$. From these eigenvalues we get that the recurrence relation is*

$$s_n = (i + 1)s_{n-1} - is_{n-2},$$

as before. Let $s_0 = 2$ and $s_1 = i + 1$. The terms of the sequence are :

$$2, i + 1, 0, 1 - i, 2, i + 1, 0, 1 - i, 2, i + 1, 0, 1 - i, \ldots.$$

This sequence has period 4, which is expected since $\operatorname{lcm}(4, 1) = 4$.

4. Open Questions

Apart from looking at second order linear recurrence relations, one can also consider looking at periodicity of sequences satisfying a third-order recurrence relation, which follows the form

$$s_n = c_1 s_{n-1} + c_2 s_{n-2} + c_3 s_{n-3}.$$

Can we obtain a sequence of period k where $k \in \mathbb{Z}$? If we can, is it possible to construct a sequence of only real numbers or are complex numbers necessary? It is our hope that the theorems discussed in this paper can be adapted to help answer these questions.

One can also think about these questions when looking at periodicity mod m, where only the last digit of the number is considered. In the case where m is prime is understood [3], see "Recursive Sequences Modulo p^2". [4]

References

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